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# Higher-dimensional extensions of Pauli spin matrices

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**Abstract.** The principle of basis set representation in terms of coordinate interchange matrices, of which the Pauli spin matrices are an example in two dimensions, are extended to three and four dimensions. The four-dimensional basis set of coordinate interchange matrices satisfies the usual conditions of completeness, but the three-dimensional basis set cannot be complete under any circumstances and an 'anticomplete' property is assigned to it. The coefficients of the basis set, when used to represent an arbitrary matrix, form a Hadamard transform of the cyclically interchanged arbitrary matrix.

## 1. Introduction

The usefulness of Pauli spin matrices need hardly be emphasised. Besides being of fundamental importance in quantum mechanics their use has been extended to optical problems as well (Jones 1948, O'Neil 1963, Whitney 1971). However, Pauli spin matrices are fundamentally two-dimensional operators. Certainly their use in three-dimensional problems has not occurred as a mathematical event and their use in four dimensions is uncommon. There has, however, been some anticipation of the four-dimensional case (Wiener 1928) and its use in optical problems has been suggested (Weeks 1934, Stephany 1975). The difficulty with the three-dimensional representation is that, as will be shown, it is impossible to represent an arbitrary matrix in terms of a complete set of nine coordinate interchange matrices. A coordinate interchange matrix is one in which all elements of any column or row are zero except for one element which is either plus or minus one, which is a special case of a coherency matrix. It is possible, however, to select a set of three-dimensional coordinate interchange matrices that are linearly independent and can therefore represent an arbitrary matrix. It is not necessary that the set be complete with respect to multiplication since it is then possible to represent the product of any two matrices within the set in terms of a linear combination of matrices within the set. Thus the property of linear independence is more important in a representation set of matrices than the property of completeness. It follows that if symmetrical linear independence of the product is invoked, that is, the matrix of coefficients of expansion of all the products of the basis set has the same number and preferably the same form, then it is impossible for the basis set to contain the unit matrix, and it is also impossible for the basis set to contain any simple matrix product of two arbitrary matrices within the set. Such a property will be defined as 'anticomplete'.

The representation of an arbitrary three-dimensional matrix in this form is apparently new, owing to the lack of recognition of the importance of basis sets that do not satisfy the definition of completeness. The completeness property of the set is fundamentally a property of convenience to the mathematician; that is, it is easier to handle a set of matrices in which the product of any two lies within the set rather than a set in which the product is a linear combination of elements of the set. If the mathematician should choose a modern digital computer to do his support work, then the completeness property makes little sense at all, since the computer can derive the linear combination of the set of matrices necessary to represent a product once, and then use 'look-up table' techniques to apply them to a given problem. Their use cannot be considered to be limited to computer problems, however. Just as Pauli spin matrices bring valuable insight into certain problems of physics, the anticomplete sets can bring similar insight.

In the following, basis set representation is made in three and four dimensions as more of an illustration of technique than as an application. Extensions to higher dimensions may then take the form of the principles utilised for the lower-dimensional cases. It should also be noted that, in order to illustrate fully the basis set representation of the Pauli spin matrix, the imaginary factor was dropped. This is not considered fundamental in any sense and, as should be obvious, the three- and four-dimensional representations can be made similarly complex. However, this author believes that the inclusion of an imaginary factor only clouds the issue.

## 2. Pauli spin matrices as two-dimensional coordinate interchange matrices

The Pauli spin matrices together with the two-dimensional unit matrix are usually written as

$$\mathbf{1}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\qquad \delta_x=\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix},\qquad \delta_y=\begin{pmatrix}0 & -\mathrm{i}\\\mathrm{i} & 0\end{pmatrix},\qquad \delta_z=\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}. \tag{1}$$

By extracting a factor of $-\mathrm{i}$, it is possible to rewrite this set in terms of a new notation:

$$\begin{aligned}\mathrm{II}_{11}&=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} &\qquad \mathrm{II}_{12}&=\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\\ \mathrm{II}_{21}&=\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} &\qquad \mathrm{II}_{22}&=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\end{aligned} \tag{2}$$

in which any arbitrary matrix ($\mathbf{a}$) can be represented as a linear combination of elements of equation (2) as

$$(\mathbf{a})=\sum_{i,j=1}^{2}\alpha_{ij}\mathrm{II}_{ij} \tag{3}$$

where the $\alpha_{ij}$ are the coefficients of the expansion. This last expression may in turn be written in matrix form as

$$\begin{pmatrix}a_{11} & a_{12}\\a_{22} & a_{21}\end{pmatrix}=\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}\alpha_{11} & \alpha_{12}\\\alpha_{21} & \alpha_{22}\end{pmatrix}. \tag{4}$$

Note that, despite its almost trivial simplicity, an arbitrary matrix is the Hadamard transform of the matrix of coefficients. Note also that the arbitrary matrix is cyclically interchanged in the second line. The significance of these two observations may become clearer later. The inverse of equation (4) above is immediately written, since the Hadamard matrix is its own inverse:

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{22} & a_{21} \end{pmatrix}. \tag{5}$$

The set of equation (2) above is complete in the normal sense of matrix multiplication and the product table is given below.

**Table 1.** Product of $\mathrm{II}_{ij}$ and $\mathrm{II}_{kl}$.

| $ij$ \ $kl$ | 11 | 12 | 21 | 22 |
|---|---|---|---|---|
| 11 | 11 | 12 | 21 | 22 |
| 12 | 12 | 11 | −22 | −21 |
| 21 | 21 | 22 | 11 | 12 |
| 22 | 22 | −22 | −12 | −11 |

By formulating the Pauli spin matrices combined with the unit matrix in just this form, it is possible to expand these results to higher dimensions.

## 3. The four-dimensional coordinate interchange basis set

The three-dimensional expansion of the above will be set aside temporarily and the four-dimensional coordinate interchange basis set will now be considered. The reason for this is ultimately the difficulty of writing the Hadamard transform in three dimensions. The four-dimensional coordinate interchange matrices are

$$\begin{gathered}
\mathrm{IV}_{11}=\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \qquad \mathrm{IV}_{12}=\begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix},\\
\mathrm{IV}_{13}=\begin{pmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{pmatrix}, \qquad \mathrm{IV}_{14}=\begin{pmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{pmatrix},\\
\mathrm{IV}_{21}=\begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix}, \qquad \mathrm{IV}_{22}=\begin{pmatrix} 0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0 \end{pmatrix},\\
\mathrm{IV}_{23}=\begin{pmatrix} 0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0 \end{pmatrix}, \qquad \mathrm{IV}_{24}=\begin{pmatrix} 0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0 \end{pmatrix},
\end{gathered} \tag{6}$$

$$
\mathrm{IV}_{31}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\qquad
\mathrm{IV}_{32}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix},
$$
$$
\mathrm{IV}_{33}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix},\qquad
\mathrm{IV}_{34}=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix};
$$
$$
\mathrm{IV}_{41}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix},\qquad
\mathrm{IV}_{42}=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix},
$$
$$
\mathrm{IV}_{43}=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix},\qquad
\mathrm{IV}_{44}=\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix},
\tag{6}
$$

Any arbitrary matrix $a_{ij}$ can be represented in terms of this set by

$$
(\mathbf{a})=\sum_{i,j=1}^{4}\alpha_{ij}\mathrm{IV}_{ij}
\tag{7}
$$

where the $\alpha_{ij}$ are the coefficients of the expansion which may be found from

$$
\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}&\alpha_{14}\\\alpha_{21}&\alpha_{22}&\alpha_{23}&\alpha_{24}\\\alpha_{31}&\alpha_{32}&\alpha_{33}&\alpha_{34}\\\alpha_{41}&\alpha_{42}&\alpha_{43}&\alpha_{44}\end{pmatrix}
=\tfrac{1}{4}\begin{pmatrix}1&1&1\\1&-1&1\\1&1&-1\\1&-1&-1\end{pmatrix}
\begin{pmatrix}1&a_{11}&a_{12}&a_{13}&a_{14}\\-1&a_{22}&a_{21}&a_{24}&a_{23}\\-1&a_{33}&a_{34}&a_{31}&a_{32}\\1&a_{44}&a_{43}&a_{42}&a_{41}\end{pmatrix}.
\tag{8}
$$

The unique properties of this set in representing polarised light have already been published (Stephany 1975) and the interesting features of the operators will not be further discussed here. The product table of these matrices are given in table 2. The products are always within the set and the set is complete with respect to matrix multiplication.

## 4. The three-dimensional coordinate interchange basis set

There is no set of nine coordinate interchange matrices that satisfies the properties illustrated by the four-dimensional basis set. Specifically, it is impossible to write a complete set of nine $3\times3$ coordinate interchange matrices that possess the property of completeness. It is possible to write a set of twelve $3\times3$ coordinate interchange matrices that are complete, but in representing an arbitrary matrix there are twelve coefficients in the expansion while only nine are needed. Equating three of the

**Table 2.** Product of $\mathrm{IV}_{ij}\mathrm{IV}_{kl}$.

| $ij$ \ $kl$ | 11 | 12 | 13 | 14 | 21 | 22 | 23 | 24 | 31 | 32 | 33 | 34 | 41 | 42 | 43 | 44 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 11 | 11 | 12 | 13 | 14 | 21 | 22 | 23 | 24 | 31 | 32 | 33 | 34 | 41 | 42 | 43 | 44 |
| 12 | 12 | 11 | 14 | 13 | −22 | −21 | −24 | −23 | 32 | 31 | 34 | 33 | −42 | −41 | −44 | −43 |
| 13 | 13 | 14 | 11 | 12 | 23 | 24 | 21 | 22 | −33 | −34 | −31 | −32 | −43 | −44 | −41 | −42 |
| 14 | 14 | 13 | 12 | 11 | −24 | −23 | −22 | −21 | −34 | −33 | −32 | −31 | 44 | 43 | 42 | 41 |
| 21 | 21 | 22 | 23 | 24 | 11 | 12 | 13 | 14 | 41 | 42 | 43 | 44 | 31 | 32 | 33 | 34 |
| 22 | 22 | 21 | 24 | 23 | −12 | −11 | −14 | −13 | 42 | 41 | 44 | 43 | −32 | −31 | −34 | −33 |
| 23 | 23 | 24 | 21 | 22 | 13 | 14 | 11 | 12 | −43 | −44 | −41 | −42 | −33 | −34 | −31 | −32 |
| 24 | 24 | 23 | 22 | 21 | −14 | −13 | −12 | −11 | −44 | −43 | −42 | −41 | 34 | 33 | 32 | 31 |
| 31 | 31 | 32 | 33 | 34 | 41 | 42 | 43 | 44 | 11 | 12 | 13 | 14 | 21 | 22 | 23 | 24 |
| 32 | 32 | 31 | 34 | 33 | −42 | −41 | −44 | −43 | 12 | 11 | 14 | 13 | −22 | −21 | −24 | −23 |
| 33 | 33 | 34 | 31 | 32 | 43 | 44 | 41 | 42 | −13 | −14 | −11 | −12 | −23 | −24 | −21 | −22 |
| 34 | 34 | 33 | 32 | 31 | −44 | −43 | −42 | −41 | −14 | −13 | −12 | −11 | 24 | 23 | 22 | 21 |
| 41 | 41 | 42 | 43 | 44 | 31 | 32 | 33 | 34 | 21 | 22 | 23 | 24 | 11 | 12 | 13 | 14 |
| 42 | 42 | 41 | 44 | 43 | −32 | −31 | −34 | −33 | 22 | 21 | 24 | 23 | −12 | −11 | −14 | −13 |
| 43 | 43 | 44 | 41 | 42 | 33 | 34 | 31 | 32 | −23 | −24 | −21 | −22 | −13 | −14 | −11 | −12 |
| 44 | 44 | 43 | 42 | 41 | −34 | −33 | −32 | −31 | −24 | −23 | −22 | −21 | 14 | 13 | 12 | 11 |

coefficients to zero accomplishes the task of the representation but is physically and aesthetically weak in that the result lacks the natural symmetry that was present in the above two- and four-dimensional representations.

The one factor that makes such a representation possible in three dimensions with nine matrices in the basis set as mentioned in the Introduction is that the product of any two matrices within the basis set can be made to be a linear combination of matrices within the basis set. This property is derived from the independence of the basis set. But to preserve the symmetrical properties it is essential that the linear combination of basis matrices used to represent the product be the same in number for the product of any two matrices within the basis set. It is this property that results in the antiset properties.

First of all, it is apparent that the desired set of nine $3\times3$ basis matrices of the coordinate interchange type cannot be complete, nor can any subset be complete. Then the product of two arbitrarily selected matrices must be composed of at least two or more other matrices within the set. To preserve the symmetry of the set, all products must be composed of the same number of linear combinations of matrices within the set. Since the unit matrix does not satisfy this property it cannot be within the set. Similarly, any of the following matrices cannot be within the set

$$\begin{pmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \\ \pm1 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & \pm1 \\ \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \end{pmatrix} \tag{9}$$

which is the set of twelve complete $3\times3$ matrices. This reasoning leads to the following set of basis matrices which is the only possible such set in three dimensions, barring sign variations, with the properties of basis set representation of the kind being sought. They are

$$\mathrm{III}_{11}=\begin{pmatrix}0&0&-1\\0&1&0\\1&0&0\end{pmatrix}\qquad \mathrm{III}_{12}=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}\qquad \mathrm{III}_{13}=\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

$$\mathrm{III}_{21}=\begin{pmatrix}0&0&1\\0&-1&0\\1&0&0\end{pmatrix}\qquad \mathrm{III}_{22}=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}\qquad \mathrm{III}_{23}=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}\qquad (10)$$

$$\mathrm{III}_{31}=\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix}\qquad \mathrm{III}_{32}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}\qquad \mathrm{III}_{33}=\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}$$

As above, any arbitrary $(\mathbf{a})$ can be represented in terms of this basis set by

$$(\mathbf{a})=\sum_{i,j=1}^{3}\alpha_{ij}\mathrm{III}_{ij} \qquad (11)$$

with the relationship of the coefficients being given by

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{22}&a_{23}&a_{21}\\a_{33}&a_{31}&a_{32}\end{pmatrix}=\begin{pmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{pmatrix}\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}\end{pmatrix} \qquad (12)$$

in which the pseudo-Hadamard matrix is not its own inverse. The inverse relationship is

$$\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}\end{pmatrix}=\tfrac{1}{2}\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{22}&a_{23}&a_{21}\\a_{33}&a_{31}&a_{33}\end{pmatrix} \qquad (13)$$

Using this last relationship to express the product of any two matrices arbitrarily selected from the basis set gives the results of table 3. Here, the value of the $\alpha$ matrix is given for the special case

$$(\mathbf{a})=\mathrm{III}_{ij}\mathrm{III}_{kl}. \qquad (14)$$

It is noted that in this table of matrices each one consists of six components only, therefore satisfying the symmetry requirement making all the matrices of the basis set of equal preference. Had a basis set of nine matrices been constructed from the set given in equation (9) above, the result would be an $\alpha$ matrix of the product of value of either one or three nonzero elements, resulting in a serious asymmetrical property within the set. The basis set of equation (10), therefore, satisfies the property of being a three-dimensional equivalent of the Pauli spin matrices.

## 5. Extensions to higher dimensions

Representation of arbitrary matrices in higher dimensions is facilitated by the following observations. First of all, let $(N_{ij})$ be a set of $N\times N$ coordinate interchange matrices in dimension $N$. Consistent with the preceding notation let $(\alpha_{ij})$ be the set of $N^2$ coefficients and $(a_{ij})$ an arbitrary matrix. Here $N$ may be equal to $2^c$, for $c$ integral, in which case $(N_{ij})$ is an ordinary complete set of basis matrices. Otherwise $(N_{ij})$ is

**Table 3.** Table for the product of $\mathrm{III}_{ij}$ and $\mathrm{III}_{kl}$.

| $ij \backslash kl$ | 11 | 12 | 13 | 21 | 22 | 23 | 31 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|---|
| 11 | $\begin{matrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{matrix}$ | $\begin{matrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{matrix}$ | $\begin{matrix} -1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{matrix}$ | $\begin{matrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \end{matrix}$ | $\begin{matrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{matrix}$ | $\begin{matrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{matrix}$ | $\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}$ | $\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$ | $\begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$ |
| 12 | $\begin{matrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{matrix}$ | $\begin{matrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \end{matrix}$ | $\begin{matrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{matrix}$ | $\begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$ | $\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}$ | $\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$ | $\begin{matrix} -1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{matrix}$ | $\begin{matrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{matrix}$ | $\begin{matrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{matrix}$ |
| 13 | $\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$ | $\begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$ | $\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}$ | $\begin{matrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{matrix}$ | $\begin{matrix} -1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{matrix}$ | $\begin{matrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{matrix}$ | $\begin{matrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{matrix}$ | $\begin{matrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{matrix}$ | $\begin{matrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \end{matrix}$ |
| 21 | $\begin{matrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{matrix}$ | $\begin{matrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{matrix}$ | $\begin{matrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{matrix}$ | $\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}$ | $\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$ | $\begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$ | $\begin{matrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \end{matrix}$ | $\begin{matrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{matrix}$ | $\begin{matrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{matrix}$ |
| 22 | $\begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$ | $\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}$ | $\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$ | $\begin{matrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{matrix}$ | $\begin{matrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \end{matrix}$ | $\begin{matrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{matrix}$ | $\begin{matrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{matrix}$ | $\begin{matrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{matrix}$ | $\begin{matrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{matrix}$ |
| 23 | $\begin{matrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{matrix}$ | $\begin{matrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{matrix}$ | $\begin{matrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \end{matrix}$ | $\begin{matrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{matrix}$ | $\begin{matrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{matrix}$ | $\begin{matrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{matrix}$ | $\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$ | $\begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$ | $\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}$ |
| 31 | $\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}$ | $\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$ | $\begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$ | $\begin{matrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{matrix}$ | $\begin{matrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{matrix}$ | $\begin{matrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{matrix}$ | $\begin{matrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{matrix}$ | $\begin{matrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{matrix}$ | $\begin{matrix} -1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{matrix}$ |
| 32 | $\begin{matrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{matrix}$ | $\begin{matrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{matrix}$ | $\begin{matrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{matrix}$ | $\begin{matrix} -1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{matrix}$ | $\begin{matrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{matrix}$ | $\begin{matrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{matrix}$ | $\begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$ | $\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}$ | $\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$ |
| 33 | $\begin{matrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{matrix}$ | $\begin{matrix} -1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{matrix}$ | $\begin{matrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{matrix}$ | $\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$ | $\begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$ | $\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}$ | $\begin{matrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{matrix}$ | $\begin{matrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{matrix}$ | $\begin{matrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{matrix}$ |

anticomplete in the sense of the above. The relationships

$$a = \mathbf{H}_N \alpha \qquad \text{and} \qquad \alpha = (1/M)\mathbf{H}_N^{-1} a \tag{15}$$

are obvious, where $\mathbf{H}_N$ is the Hadamard matrix for $2^c$, $c$ still being integral, and $\tilde{\mathbf{H}}_N \mathbf{H}_N \neq 1$ for $c$ not integral. $M$ is a unique normalisation factor. To facilitate finding the representations written in equation (15) above, $a$ and $\alpha$ may be rewritten in vector form as $\boldsymbol{a}$ and $\boldsymbol{\alpha}$. In general,

$$\boldsymbol{a} = (a_{11}a_{12} \ldots a_{1N}a_{21} \ldots a_{2N} \ldots a_{NN}) \tag{16}$$

with a similar relationship $\boldsymbol{\alpha}$. It is now possible to write the transforms of equation (15) as

$$\boldsymbol{a} = \mathbf{H}_{N^2} \boldsymbol{\alpha} \tag{17a}$$

and

$$\boldsymbol{\alpha} = (1/M^2)\mathbf{H}_{N^2}^{-1} a. \tag{17b}$$

The second relationship will always exist for some basis set, provided $\mathbf{H}_{N^2}$ is not singular. It will not be singular if the basis set is linearly independent. The general form for $\mathbf{H}_{N^2}$ is

$$\mathbf{H}_{N^2} = \begin{pmatrix} (N_{11})_{11} & (N_{12})_{11} & \ldots & (N_{NN})_{11} \\ (N_{11})_{12} & (N_{12})_{12} & & (N_{NN})_{12} \\ \vdots & \vdots & & \vdots \\ (N_{11})_{1N} & (N_{12})_{1N} & & (N_{NN})_{1N} \\ (N_{11})_{21} & (N_{12})_{21} & & (N_{NN})_{21} \\ \vdots & \vdots & & \vdots \\ (N_{11})_{NN} & (N_{12})_{NN} & \ldots & (N_{NN})_{NN} \end{pmatrix} \tag{18}$$

The inverse may be found immediately from the relationship

$$\mathbf{H}_{N^2}^{-1}\mathbf{H}_{N^2} = 1. \tag{19}$$

The only problem remaining is the reduction back to the $N$th dimension from the $N^2$th dimension. This reduction can be accomplished by interchanging vertical columns of the matrix until the submatrix of any $N \times N$ dimensional matrix has all off-diagonal elements equal to zero. This procedure explains the interchanging of the $(a_{ij})$ elements in equations (4), (5), (8), (12) and (13). It is also clear that this interchanging can always be accomplished because of the nature of the coordinate interchange definition. Furthermore, each $N \times N$ submatrix is diagonal and will have all elements on the diagonal equal. If the elements of this so interchanged matrix are designated $h_{ij}$, then

$$\underline{\mathbf{H}}_{N^2}^{-1} = \begin{pmatrix} h_{11}1_N & h_{12}1_N & \ldots & h_{1N}1_N \\ \vdots & & & \vdots \\ h_{N1}1_N & h_{N2}1_N & \ldots & h_{NN}1_N \end{pmatrix} \tag{20}$$

where the bar under the $\mathbf{H}$ indicates that the matrix has been so cyclically interchanged and $1_N$ is the $N$-dimensional unit vector. It is now possible to write the expression for $\mathbf{H}_N^{-1}$:

$$\mathbf{H}_N^{-1} = \begin{pmatrix} h_{11} & \dots & h_{1N} \\ \vdots & & \vdots \\ h_{N1} & \dots & h_{NN} \end{pmatrix}. \tag{21}$$

This solves the case in general and enables the writing of transforms for any independent set of coordinate interchange basis matrices in any number of dimensions.

## 6. Conclusions

A set of nine $3 \times 3$ matrices has been found having a similar function to the Pauli spin matrices in representing matrix operators. A like set, which has already been used in optics, has also been presented. Extensions to higher dimensions and to a complex form is an easy task using the unique properties of the antiset. Although originally designed for the task of simplifying computer programming, use of the higher-order Pauli spin matrices also lends insight to the expression in compact form of various matrix operators. An illustration of this procedure has already been published (Stephany 1975), and applications to problems other than optical are expected.

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